Boundary-Layer Computation by an N Parameter Integral Method Using Exponentials

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A new integral method is presented for the accurate calculation of laminar boundary-layer flows. The method is written for unspecified order N of approximation and permits rapid qualitative solutions for low N (1 or 2), and very accurate solutions for high N (3 to 5). Exponentials are used in the velocity approximation and as weighting functions. The resulting integral relations and a simple method for obtaining initial profile parameters are given. Results are presented for flow with continuous and discontinuous external pressure gradient and suction conditions and for favorable and adverse pressure gradients covering the spectrum of boundary-layer flows from stagnation point to separation. For $N = 5$ the absolute error in the wall shear is generally of the order of 10^{-4}.

I. Introduction

THERE are several excellent methods presently available for the accurate numerical calculation of plane laminar boundary-layer flows. To name the most prominent: implicit finite difference methods (Blottner and Flügge-Lotz,[1] Schönauer,[2] Krause[3]), the difference-differential scheme of the Hartree-Womersley[4] method (Smith and Clutter[5]), and the Dorodnitsyn[6] integral method, whose potential has been thoroughly explored by Bethel.[7,8] The latter investigation showed that integral methods (methods of "weighted residuals") can produce excellent results at a considerable saving in computing expense. The present paper will describe another such method.

Integral formulations for two-dimensional problems reduce the problem to the solution of a set of a few ordinary differential equations, do not require iteration, pose no stability problems and promise a corresponding reduction in computing time by an order of magnitude compared to difference and difference-differential schemes. These advantages are realized in the Dorodnitsyn method. However, this method has a few undesirable shortcomings: different approximating functions are required for different types of flow; velocity overshoot cannot be handled; numerical quadrature of integrals is required for $N > 4$; some velocity profiles do not correspond to complete sets in the shear profile approximation by functions of U.

The present method was developed with a view towards these problems. An accurate method of boundary-layer calculation has been developed, and the concepts have also been applied successfully to the solution of axisymmetric wake, jet, and vortex flows and general nonadiabatic compressible boundary-layer flow. The method uses a complete set of approximating functions in Y (not U) with an unspecified number N of parameters $a_n(X)$. This permits approximation of any physically possible velocity profiles, including those with flow reversal and overshoot. With the exception of the standard numerical integration of the ordinary differential equations for the N parameters, all operations are analytical and involve only the solution of a set of linear algebraic equations at each step. At each streamwise station the velocity profile becomes available in analytic form as a short series of a few exponentials—an advantage for stability calculations.

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In Sec. II of this paper general integral relations will first be derived for the laminar boundary layer. A scaling transformation is introduced for computational reasons. Specific weighting and approximating functions are adopted in Sec. III. The general integral relations can then be integrated, yielding a set of ordinary differential equations for the parameters in the velocity approximation. These equations are given, and their solution is explained. In the case of similar flows the equations reduce to a nonlinear algebraic set for the parameters. Initial profiles are required to start the computation. These can be obtained from a solution of the similar equations, or, more generally, by a least-squares method for obtaining the initial profile parameters described in Sec. IV. Section V summarizes the computing procedure, and Sec. VI presents a variety of test cases for continuous and discontinuous, favorable and adverse pressure gradients and different suction conditions. Section VII summarizes the results.

II. General Formulation

With the introduction of a Reynolds number and the usual transformations

$$X = x/l,\ Y = Re^{1/2}y/l,\ U = u/u_\infty,\ V = Re^{1/2}v/u_\infty$$

the laminar incompressible boundary-layer equations become nondimensional and can be written, in divergence form, as

$$(\partial U^2/\partial X) + [\partial(VU)/\partial Y] = U_e(dU_e/dX) + (\partial^2 U/\partial Y^2)$$
$$(\partial U/\partial X) + (\partial V/\partial Y) = 0 \qquad (1)$$

To maintain the same degree of computational accuracy at each X-wise station, it is desirable to stretch the Y coordinate to keep approximately constant boundary-layer thickness in the $(\bar{X},\bar{Y})$ plane of computation. A stretching function $g(X)$ is therefore introduced. The choice of $g(X)$ depends on the case, i.e., for a flat plate problem $g(X) = X^{1/2}$ would be appropriate. The new independent variables are then

$$\bar{Y} = Y/g(X),\ \bar{X} = X$$

Introduction of these into the boundary-layer Eqs. (1) suggests an additional transformation of the velocities

$$\bar{U} = Ug,\ \bar{V} = V - \bar{Y}(dg/dX)U$$

The transformed boundary-layer equations now become

$$[\partial(\bar{U}^2)/\partial\bar{X}] + [\partial(\bar{U}\bar{V})/\partial\bar{Y}] = [\bar{U}^2(dg/dX)/g] + g^2U_e(dU_e/dX) + (1/g)(\partial^2\bar{U}/\partial\bar{Y}^2) \qquad (2)$$
$$(\partial\bar{U}/\partial\bar{X}) + (\partial\bar{V}/\partial\bar{Y}) = 0$$

with the boundary conditions

$$\bar{U}(\bar{X},0) = 0,\ \bar{U}(\bar{X},\infty) = U_e(X)g(X)$$

$$\bar{V}(\bar{X},0) = V_0(X),\ \bar{U}(0,\bar{Y}) = U(0,Y)g(0) \qquad \text{(initial profile)}$$

The method of weighted residuals will be used to reduce the partial differential equations (2) in $\bar{X}$ and $\bar{Y}$ to a set of ordinary differential equations in $\bar{X}$. The $\bar{Y}$-dependence is eliminated by assuming the functional form of the velocity profile $U[\bar{Y},a_n(\bar{X})]$ in the $\bar{Y}$ direction incorporating the correct behavior at the boundaries and employing N free parameters $a_n(\bar{X})$, and integrating the momentum equation in the $\bar{Y}$ direction. The system of N linearly independent equations for the $a_n(\bar{X})$ is generated by multiplying the momentum equation by N members of a set of linearly independent weighting functions $f_k(\bar{Y})$ before integration. Integration is over the semi-infinite domain $0 \leq \bar{Y} < \infty$. Convergence of the integrals requires $f_k(Y)$ = finite everywhere, and $f_k(\infty) = 0$. The integral relations for the plane laminar incompressible boundary layer become

$$\frac{d}{d\bar{X}}\int_0^\infty f_k\bar{U}^2d\bar{Y} - \int_0^\infty f_k'\bar{V}\bar{U}d\bar{Y} - \frac{1}{g}\int_0^\infty f_k''\bar{U}d\bar{Y} +$$
$$\bar{T}_0 - U_e\frac{dU_e}{dX}g^2\int_0^\infty {}_k d\bar{Y} - \frac{dg/dX}{g}\int_0^\infty f_k\bar{U}^2d\bar{Y} = 0 \qquad (3)$$
$$k = 1,2,\ldots,N$$

where

$$\bar{T}_0 = \left[\frac{f_k}{g}\frac{\partial\bar{U}}{\partial\bar{Y}}\right]_{\bar{Y}=0}$$

in the boundary-layer case, and

$$\bar{T}_0 = -\left[\frac{f_k'\bar{U}}{g}\right]_{\bar{Y}=0}$$

in the case of the plane jet or wake. Integration of the continuity equation permits replacement of $\bar{V}(\bar{X},\bar{Y})$ in Eq. (3):

$$\bar{V}(\bar{X},\bar{Y}) = -\int_0^{\bar{Y}}\frac{\partial\bar{U}}{\partial\bar{X}}d\bar{Y} + V_0(X)$$

$V_0(X)$ is the prescribed suction or blowing velocity.

Particular choices of weighting functions $f_k(\bar{Y})$ and velocity approximation $\bar{U}[\bar{Y},a_n(\bar{X})]$ must now be made to permit integration of Eq. (3) with respect to $\bar{Y}$.

III. Specific Formulation

The chosen weighting and approximating functions must be at least linearly independent and must result in convergent integrals. A necessary (but generally not sufficient) requirement for convergence of the integral method is the choice of a complete set of approximating functions. Analytical integrability of the integrals is not essential, but will be required here to increase speed and accuracy of the computation. The approximating expression for $\bar{U}(\bar{X},\bar{Y})$ must satisfy the correct boundary conditions, and should give the exponential behavior of boundary-layer velocity profiles as $\bar{Y} \to \infty$. It should possibly contain an exact solution. Finally, the weighting functions should weight most heavily in the region where the solution is being sought, i.e., near the wall $\bar{Y} = 0$. These requirements are satisfied by using members of the set $\{e^{-\gamma\bar{Y}}\bar{Y}^m\}$, in both the weighting and approximating functions. Note that then extremely simple integrals result of the form

$$\int_0^\infty e^{-\gamma\bar{Y}}\bar{Y}^m d\bar{Y} = \frac{m!}{\gamma^{m+1}}$$

The particular functions chosen for the solution of the boundary-layer problem are as follows:

Weighting functions

$$f_k(\bar{Y}) = e^{-\sigma_k\bar{Y}},\ k = 1,2,\ldots,N \qquad (4)$$

Velocity approximation

$$\bar{U}(\bar{X},\bar{Y}) = (1 - e^{-\alpha\bar{Y}})\left[U_e(\bar{X}) + \sum_{n=1}^{N}\bar{a}_n(\bar{X})e^{-n\alpha\bar{Y}}\right] \qquad (5)$$

α is a constant and should correspond roughly to the exponential decay of the velocity profile in the zero-order approximation. Note that the formulation (5) satisfies the boundary conditions and contains the asymptotic suction profile as an exact solution when $a_n = 0,\ n \geq 1$. The general asymptotic behavior of boundary-layer velocity profiles would suggest an expansion using exponentials in $\bar{Y}^2$. However, difficulties then arise in the analytical (closed form) integration of the integral relations.

The transformation $e^{-\alpha\bar{Y}} = \zeta$ gives some insight into the properties of the approximation (5). For fixed $\bar{X}$, relation (5) becomes a polynomial expression in ζ in the finite domain $1 \leq \zeta < 0$ corresponding to $0 \leq \bar{Y} < \infty$. Convergence of the approximation for any continuous $\bar{U}$ is then guaranteed by the Weierstrass approximation theorem. However, convergence of the integral method does not necessarily follow. The orthonormal set corresponding to the present problem is the set of shifted Legendre polynomials in ζ, but it will not be used in the present work.

Substitution of weighting functions (4) and velocity approximation (5) into the integral relations (3) leads to N ordinary differential equations of the form

$$\sum_{n=1}^{N} C_{n,k}\frac{d\bar{a}_n}{dX} = D_k,\ k = 1,2,\ldots,N \qquad (6)$$

where

$$C_{n,k} = \sum_{l=0}^{N}\bar{a}_l P(k,l,n)$$

$$D_k = \frac{\dot{U}_eU_eg^2}{\sigma_k} - \frac{\alpha}{g}\sum_{l=0}^{N}\bar{a}_l - (\dot{U}_eg + U_e\dot{g})\sum_{l=0}^{N}\bar{a}_lQ(k,l) -$$
$$\left(V_0\sigma_k - \frac{\sigma_k^2}{g}\right)\sum_{l=0}^{N}\bar{a}_lR(k,l) + \frac{\dot{g}}{g}\sum_{n=0}^{N}\sum_{l=0}^{N}\bar{a}_n\bar{a}_lS(k,l,n) \qquad (7)$$

with $\bar{a}_0(X) = \bar{U}_e(X)$ and d/dx represented by a dot. P, Q, R, and S are numbers which are determined only once at the beginning of the calculation.

$$P(k,l,n) = [\sigma_k + (n+l)\alpha]^{-1} - 4[\sigma_k + (n+l+1)\alpha]^{-1} +$$
$$2[\sigma_k + (n+l+2)\alpha]^{-1} + \frac{\sigma_k}{n\alpha}\{[\sigma_k + (n+l)\alpha]^{-1} -$$
$$[\sigma_k + l\alpha]^{-1} - [\sigma_k + (n+l+1)\alpha]^{-1} +$$
$$[\sigma_k + (l+1)\alpha]^{-1}\} + \frac{\sigma_k}{(n+1)\alpha}\{[\sigma_k + l\alpha]^{-1} -$$
$$[\sigma_k + (n+l+1)\alpha]^{-1} + [\sigma_k + (n+l+2)\alpha]^{-1} -$$
$$[\sigma_k + (l+1)\alpha]^{-1}\}$$

$$Q(k,l) = 2[\sigma_k + l\alpha]^{-1} - 4[\sigma_k + (l+1)\alpha]^{-1} +$$
$$2[\sigma_k + (l+2)\alpha]^{-1} + \frac{\sigma_k}{\sigma_k + l\alpha}\{\alpha^{-1} - [\sigma_k + l\alpha]^{-1}\} + \qquad (8)$$
$$\frac{\sigma_k}{\sigma_k + (l+1)\alpha}\{[\sigma_k + (l+1)\alpha]^{-1} - 2\alpha^{-1}\} +$$
$$\frac{\sigma_k}{\alpha}[\sigma_k + (l+2)\alpha]^{-1}$$

$$R(k,l) = [\sigma_k + l\alpha]^{-1} - [\sigma_k + (l+1)\alpha]^{-1}$$

$$S(k,l,n) = [\sigma_k + (n+l)\alpha]^{-1} - 2[\sigma_k +$$
$$(n+l+1)\alpha]^{-1} + [\sigma_k + (n+l+2)\alpha]^{-1}$$

The system (6) of first order ordinary differential equations for the $d\bar{a}_n/dX$ is solved by standard methods. The resulting $\bar{a}_n$ are used to calculate the velocity profile, wall shear, displacement thickness, and momentum thickness in the nondimensional physical (X,Y) plane. Let $a_n^* = \bar{a}_n/(gU_e)$. Then nondimensional velocity

$$U(X,Y)/U_e(X) = (1 - e^{-\alpha Y/g}) \times \left(1 + \sum_{n=1}^{N} a_n^*(X)e^{-n\alpha Y/g}\right) \quad (9)$$

nondimensional wall shear stress

$$T_w = \left.\frac{\partial U}{\partial Y}\right|_{Y=0} = \frac{\alpha U_e}{g}\left(1 + \sum_{n=1}^{N} a_n^*\right) \quad (10)$$

nondimensional displacement thickness

$$\Delta_1 = \int_0^\infty \left(1 - \frac{U}{U_e}\right)dY = \frac{g}{\alpha}\left[1 + \sum_{n=1}^{N} a_n^*\left(\frac{1}{n+1} - \frac{1}{n}\right)\right] \quad (11)$$

nondimensional momentum thickness

$$\Delta_2 = \int_0^\infty \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)dY = \frac{g}{\alpha}\left[\frac{1}{2} + \sum_{n=1}^{N} a_n^*\left(\frac{3}{n+1} - \frac{2}{n+2} - \frac{1}{n}\right) + \sum_{n=1}^{N}\sum_{l=1}^{N} a_n^*a_l^*\left(\frac{2}{n+l+1} - \frac{1}{n+l+2} - \frac{1}{n+l}\right)\right] \quad (12)$$

Similar Flows

In the case of similarity the normalized velocity profile $U(X,Y)/U_e(X)$ remains an unchanged function of the similarity variable $\bar{Y} = Y/g(X)$ for all X. Derivatives of the parameters $a_n^*(X)$ in the velocity expression (9) are then zero. Noting that $\bar{a}_n = a_n^* g U_e$, the equations for similar profiles can be derived from Eqs. (6) and (7) by letting $da_n^*/dX = 0$

$$\frac{\dot{U}_e g^2}{\sigma_k} + \sum_{l=0}^{N} a_l^*\left[\dot{g}gU_e \sum_{n=0}^{N} a_n^* S(k,l,n) - g\frac{d}{dX}(gU_e)\sum_{n=1}^{N} a_n^* P(k,l,n) - \alpha - \frac{d}{dX}(gU_e)gQ(k,l) - (V_0\sigma_k g - \sigma_k^2)R(k,l)\right] = 0 \quad (13)$$

$$k = 1,2,\ldots,N$$

After prescribing the exponent α, the external velocity $U_e(X)$, the similarity function $g(X)$, and the suction or blowing velocity $V_0(X)$ for the similarity case at hand, the set (13) of N nonlinear algebraic equations can be solved iteratively for the N parameters a_n^* of the similarity profile. The P, Q, R, S are given by Eqs. (8). For the asymptotic suction profile,

$$U_e(X) = 1,\ g(X) = 1,\ V_0(X) = -1$$

for the Blasius profile

$$U_e(X) = 1,\ g(X) = X^{1/2},\ V_0(X) = 0$$

and for the stagnation point profile

$$U_e(X) = X,\ g(X) = 1,\ V_0(X) = 0$$

Standard references may be consulted for other similarity cases.

Solutions of Eqs. (13), in particular those corresponding to stagnation point and Blasius flat plate flow, can be used as initial profiles in many nonsimilar computations.

IV. Initial Profiles

Initial velocity profiles must be supplied in a form compatible with the integral method, i.e., in the form (5). This requires either a solution of Eqs. (13) for similar profiles, or a method for accurately determining the parameters $\bar{a}_n$ in the approximation (5) for arbitrary velocity profiles. A least-squares-type method is used for this purpose.

An arbitrary tabulated velocity profile $U(Y)$ is to be approximated by

$$U_N(Y) = (1 - e^{-\alpha Y})\left(U_e + \sum_{n=1}^{N} a_n e^{-n\alpha Y}\right)$$

where α is prescribed. The N unknown coefficients a_n are determined by equating N integrals of the form

$$\int_0^\infty f_k(Y)U_N(Y)dY = \int_0^\infty f_k(Y)U(Y)dY \equiv q(k)$$

where

$$f_k(Y) = e^{-\sigma_k Y},\ k = 1,2,\ldots,N$$

The value of the last integral $q(k)$ is determined numerically, using the tabulated velocity data. N equations result for the coefficients a_n,

$$\sum_{n=1}^{N} a_n[(\sigma_k + n\alpha)^{-1} - \{\sigma_k + (n+1)\alpha\}^{-1}] = q(k) - U_e[\sigma_k^{-1} - (\sigma_k + \alpha)^{-1}],\quad k = 1,2,\ldots,N \quad (14)$$

The procedure reduces to a least-squares approximation when $\sigma_k = \alpha k$, $k = 1,2,\ldots,N$.

The parameters a_n are obtained from the linear system (14) by Gaussian elimination. To test the theoretical convergence properties of the present velocity approximation, the mean-square error

$$\int_0^\infty [U(Y) - U_N(Y)]^2 dY$$

has been computed for a ramp profile ($U = Y$ for $0 \le Y \le 1$, and $U = 1$ for $Y > 1$) as a function of order of approximation N. In this simple case no numerical integrations are required. The mean-square error decreases uniformly from 10^{-2} for $N = 1$ to 10^{-4} for $N = 9$.

Many practical boundary-layer calculations can be started by using either the Blasius, or the stagnation profile. The parameters a_n for these profiles, found by profile approximation for $N = 1$, 2, 3 and $\alpha = 1$ are given in Table 1. It has been found that for calculations with $N > 3$ initial profiles for $n = 2$ or 3 can be conveniently used without noticeable loss in accuracy.

V. Computing Procedure

An outline of the computing procedure will serve to summarize the method. For convenience, let the exponent $\alpha = 1$, and the weighting function exponents $\sigma_k = 1,2,\ldots,k,\ldots N$. These values have been used for the test cases of Sec. VI.

Table 1 Initial profile parameters

N	a_1	a_2	a_3
	Stagnation point profile ($\alpha = 1$)		
1	0.54116		
2	0.97637	−0.87041	
3	1.06658	−1.32149	0.45108
	Blasius profile ($\alpha = 1$)		
1	−1.18149		
2	−1.97934	1.59569	
3	−2.38035	3.60073	−2.00504

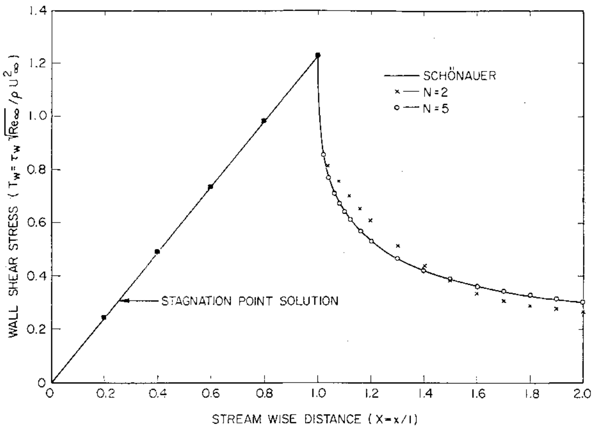

Fig. 1 Wall shear stress for discontinuous external pressure gradient.

Initial profile parameters are first obtained, either from Table 1, or from solution of Eqs. (13) for similar flows, or by numerical integration of tabulated data for $q(k)$, and the solution of system (14) for the a_n. Note that $\bar{a}_n = a_n g$. The functions $g(X)$ (roughly proportional to expected boundary-layer thickness) and $\dot{g}(X)$, and the prescribed velocity and suction distributions $[U_e(X), \dot{U}_e(X), V_0(X)]$ are introduced into the program.

At the start of the calculation, the coefficient arrays P,Q,R,S [Eqs. (8)] are first computed and stored. After that the integration of the N ordinary differential equations (6) for the $\bar{a}_n(X)$ is started with the initial profile parameters. At each X-wise step in the integration the coefficient matrices $C_{n,k}$ and D_k [Eqs. (7)] are first computed; Eq. (6) is then solved by Gaussian elimination for the $d\bar{a}_n/dX$ vector. A standard method of integration (Runge-Kutta, or predictor-corrector) then produces the $\bar{a}_n$, and the calculation advances one step in the X direction. When printout is desired, an intermediate calculation produces the velocity profile, wall shear, displacement, and momentum thicknesses from Eqs. (9–12).

VI. Applications

Results of the method will now be presented for nonsimilar boundary-layer flows incorporating continuous and discontinuous suction, and continuous and discontinuous, favorable and adverse pressure gradients. The cases presented have previously been computed by various authors by various methods, and direct comparison is possible. In all solutions $\alpha = 1$ and $\sigma_k = 1,2,\ldots,k,\ldots,N$ were used. $g(X)$ was either unity or $X^{1/2}$.

Discontinuous External Flow

Schönauer[2] used a finite-difference method to compute the abrupt change from stagnation point flow ($U = X$) to flat plate flow ($U = 1$). Smith and Clutter[5] and Bethel[7] have used this flow as a test case for the Hartree-Womersley[4] and Dorodnitsyn[6] methods, respectively. Figure 1 presents results from the present integral method for $N = 2$ and $N = 5$. The initial similar portion of the solution was used to check the behavior of the method under conditions of similarity. For $N = 5$ the values for the wall shear stress differ from those given by Schönauer by about 0.00002 at $X = 1$, 0.0015 at $X = 1.5$, and 0.006 at $X = 2$. Corresponding differences in displacement thickness are 0.004, 0.002, and 0.02, respectively. These results are more accurate than those obtained by Smith and Clutter; they compare with those obtained by Bethel. The results for $N = 2$ are accurate in the region of stagnation point flow, but differ substantially in the flat plate region.

Discontinuous Suction

The discontinuous application of suction has some technical importance. Rheinboldt[9] used a series expansion method to calculate flat plate flow with constant boundary-layer suction $V_0 = -1.5$ for $1 < X < 1.15$. Smith and Clutter[5] and Bethel[8] also applied their methods to this problem. Results from the present integral method are shown in Fig. 2 for $N = 2$ and $N = 5$. As in the case of discontinuous external pressure gradient, the discontinuous suction distribution caused no computational problems. The results for $N = 5$ are in agreement with Bethel's over the entire region, and with those by Smith and Clutter in the suction region and immediately downstream of it ($1 < X < 1.4$). On the basis of these comparisons, it is concluded that Rheinboldt's solution is in error in the region $1 < X < 1.4$. This conclusion is substantiated in a recent study by Krause.[10]

Circular Cylinder

The circular cylinder incorporates features typical of airfoil boundary-layer flows—acceleration from a stagnation point in a favorable pressure gradient, then deceleration in an adverse pressure gradient and separation. Terrill's[11] solutions for the circular cylinder ($U = \sin X$) with and without suction have been used as a standard by Schönauer,[2] Bethel,[7,8] and for the present method. Results for the circular cylinder with suction ($V_0 = 0.5$) and for $N = 2$ and $N = 5$ are plotted in Figs. 3 and 4. The results for $N = 5$ are in excellent agreement with those given by Terrill. Figure 5 shows the distribution of the absolute error in wall shear and displacement thickness as a function of X. For $N = 5$ the relative error is of the order of one hundredth of one percent for the wall shear, and less than one percent in the displacement thickness over most of the region of integration. The program failed at $X = 2.0034$ (for $N = 5$) where gradients become very large. The extrapolated separation point is at $X = 2.00355$. Terrill's result is $X_s = 2.00164$. For $N = 1$ the program integrates through the separation point into the region of separated and reversed flow.

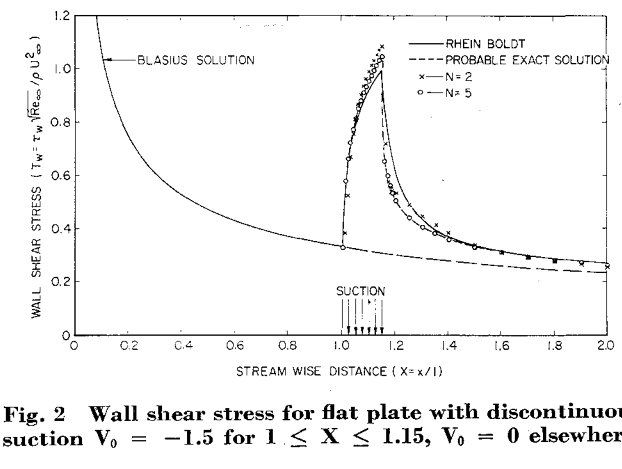

Fig. 2 Wall shear stress for flat plate with discontinuous suction $V_0 = -1.5$ for $1 \leq X \leq 1.15$, $V_0 = 0$ elsewhere.

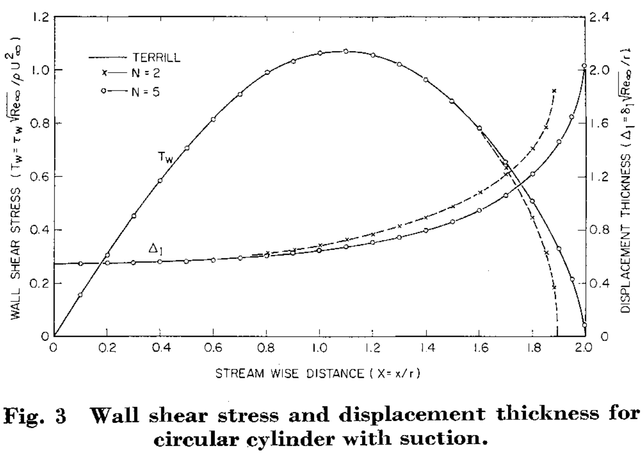

Fig. 3 Wall shear stress and displacement thickness for circular cylinder with suction.

VII. Summary and Conclusions

Using the concept of "weighted residuals" techniques an integral method has been developed and applied successfully to the accurate solution of typical boundary-layer flows. The boundary-layer equations are integrated analytically over the semi-infinite region $0 \leq Y < \infty$ by using exponentials in the weighting functions and the approximating expression for velocity $U(X,Y)$. The unknown coefficients in this expression are determined by numerical integration in the X direction. The velocity approximation itself is convergent for all possible velocity profiles, including those with overshoot or flow reversal. In contrast to the Dorodnitsyn technique[7,8] no special approximating functions are required for different types of flow. Initial profiles are obtained from a set of nonlinear algebraic equations for similar flows, or by a least-squares method requiring the solution of a linear algebraic system.

The results presented show that the method can be used both for rapid qualitative estimates ($N = 1$ or 2) or for very accurate computations N ($N = 3$ to 5). The present computing program was developed to demonstrate the feasibility of the approach, and no attempts were made to obtain greatest efficiency. Error bounds used in the integration were probably unnecessarily strict, and very small step sizes were usually used. The Runge-Kutta routine, itself slower than other integration routines, had no step doubling provisions.

For $N = 2$, all cases ran between 10 and 20 sec on the IBM 360/75. Computation times increase with the order of approximation, and with increasing pressure gradients in the external flow. Thus the most adverse case, the circular cylinder with $N = 5$, took about 10 min, with about 90% of this time spent in the stagnation region for $X < 0.2$. The increase in computing time is a direct result of larger gradients in the approximating parameters. In a similar flow, for example, the parameters $\bar{a}_n$ used in the computation are proportional to gU_e. Use of the quasi-similar velocity approximation Eq. (9) instead of Eq. (5) would eliminate this problem. Further substantial improvement is to be expected from introduction of the appropriate orthonormal set of approximating functions (i.e., the shifted Legendre polynomials in $\zeta = \exp(-\alpha\bar{Y})$). These improvements should increase efficiency by an order of magnitude.

The computation error is lowest for the wall shear stress and somewhat higher for displacement thickness and momentum thickness. The reason is the exponential weighting of the integrals, which accounts for a more accurate representation of velocity profiles near the wall. An effect of the choice of exponents α and σ_k on the accuracy of the results is therefore to be expected, but has not yet been systematically studied.

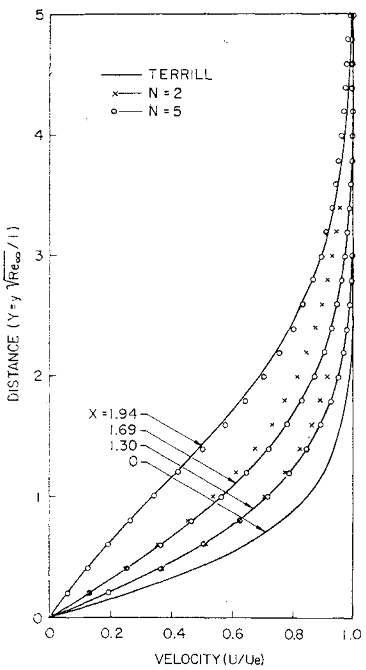

Fig. 4 Velocity profiles for circular cylinder with suction.

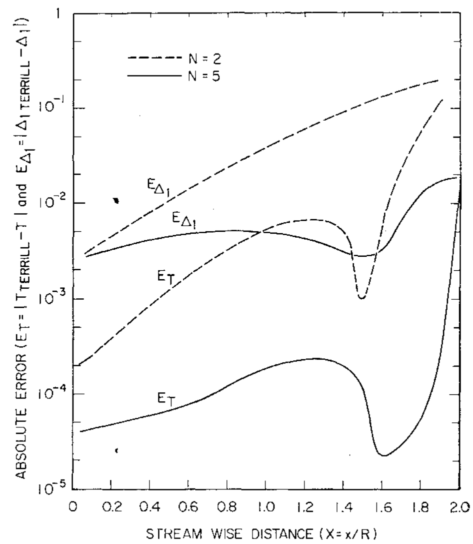

Fig. 5 Absolute error in wall shear stress and displacement thickness for circular cylinder with suction.

The concepts of the present approach have also been used to develop integral methods for the computation of arbitrary plane compressible boundary-layer flow, and of axisymmetric wake, jet, and vortex flows. These results will be reported elsewhere. In the case of the vortex and the compressible boundary layer, a second approximating expression must be introduced for circulation and enthalpy, respectively. This increases the complexity of the formulation but does not require a different approach.

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